Topics in 3-manifold topology

V5D2 Selected topics in topology

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target audience: M.Sc Math (also B.Sc., PhD, ...)

Exams → Feb/ March (oral?) → content: lectures + exercises

Questions?! Ask whenever they come up! + Question session at end of semester

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Announcements etc → my website:

https://guests.mpim-bonn.mpg.de/truoel/3-mfds.html

No class on Dec 4 (Dies academicus)

Prerequisites:

- Point set topology (connected, compact, ... top. space)
- Basic algebraic topology, e.g. covering spaces, π_1
- Singular homology (& cohom.) useful, but not strictly necessary (we will recall or black box these concepts & you should tell me if sth is not clear / you feel you need more background)
- Recall: smooth/top manifold → next week

Today: overview & examples → will be sketchy

§.1 Introduction (sketchy!)

We will study **3-manifolds**, i.e. second countable Hausdorff spaces that "locally look like" $\mathbb{R}^3$ or $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 \geq 0\}$.

We will assume that they have a smooth structure, are compact, and are usually connected and orientable/oriented

--- we'll recall def'ns later

Notation: $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, $\partial D^n = S^{n-1}$

In M^n, $n := \dim(M)$ denotes the dimension of M.

§1.1 Examples

1) $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ — favorite ex :) — cpct, conn, no ∂

$= \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$

$\cong_{C^0} \mathbb{R}^3 \cup \{\infty\}$ (one-pt compactification) (exer)

(homeo) **3-sphere**

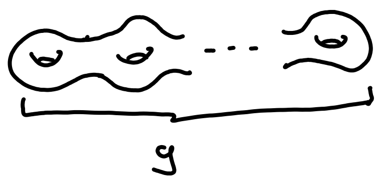
2) $D^3 = \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$ **3-ball**, $\partial D^3 = S^2$ 

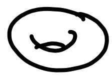
manifold w/ boundary

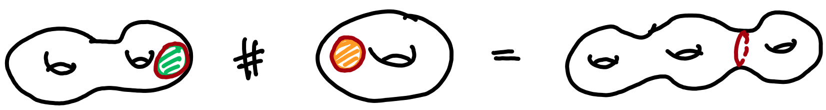
3) $\mathbb{R}^3$ non-compact :(

easy constructions of new mfds: products

4) $S^1 \times S^2$, $S^1 \times \Sigma^2_g$, where

Σ^2_g = (cld, conn.) surface of genus g = 

$\Sigma^2_0 = S^2$, $\Sigma^2_{g+1} := \Sigma^2_g \# T^2$, $T^2 = S^1 \times S^1$ 



(**connected sum**)

Choose disks $D_i \subset \mathrm{int}(\Sigma^2_i)$, remove interiors, take disjoint union of results & identify the boundaries of the disks

$\Sigma^2_g \# T^2 := (\Sigma_g \setminus \mathring{D}^2_1) \cup_{\varphi|_{\partial D^2_1}} (T^2 \setminus \mathring{D}^2_2)$, $\varphi: D^2_1 \xrightarrow{\cong_{C^\infty}} D^2_2$ orientation-reversing diffeo

Recall: $M \cup_\varphi N := M \sqcup N / x \sim \varphi(x)$

Classification of surfaces: $\forall$ closed, connected, oriented/orientable ($H_2(\Sigma) \cong \mathbb{Z}$) surfaces Σ^2 $\exists!\, g$ s.t. $\Sigma^2 \cong_{C^\infty} \Sigma^2_g$.

Rem: It's not entirely obvious, that connected sums are well defined. → later!

5) $T^3 = S^1 \times S^1 \times S^1$ 3-torus

Exer: Compute π_1 for the 3-manifolds in Ex. 1-5.

6) Lens spaces $L_{p,q} := S^3/\sim$

where $S^3 \subseteq \mathbb{C}^2$, $(z,w) \sim (e^{2\pi i/p} z, e^{2\pi i q/p} w)$,

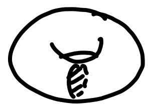
$p, q \in \mathbb{Z}$ coprime

[Covering space action of $\mathbb{Z}/p\mathbb{Z} = \pi_1(L_{p,q})$ on S^3]

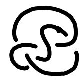
(one can make more precise when gp actions give rise to manifolds as quotients)

5') $\mathbb{Z}^3 \curvearrowright \mathbb{R}^3$ via $v \cdot w := v + w \rightsquigarrow \mathbb{R}^3/\mathbb{Z}^3 = T^3$.

7) Solid torus $V = S^1 \times D^2$, $\partial(S^1 \times D^2) = S^1 \times S^1$

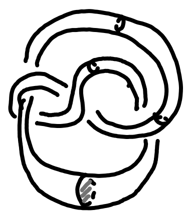


filled-in torus "donut" $V \underset{\text{h.e.}}{\simeq} S^1$ (homotopy equivalent)

8) Complements of knots $K \subseteq S^3$, e.g $K =$ 

↳ 1-dim, conn., cld (or.) submfd

Knot exterior $X_K := S^3 \setminus \operatorname{int} \underbrace{V(K)}_{\text{tubular nbhd of } K}$

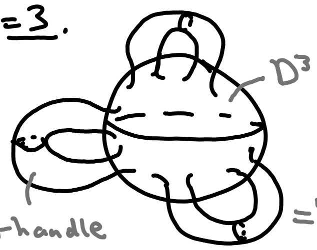


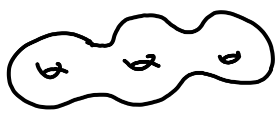
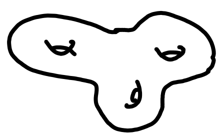
$\partial X_K = \partial V(K) \cong T^2 = S^1 \times S^1$

9) handlebody of genus g

:= "cpct 3-mfd bounded by genus g surface" = cld regular nbhd of appropriate graph

e.g $g=3$.



=  = 

= "3-ball w/ 1-handles attached"   ↳ "filled tubes"

$g=0$: D^3

$g=1$: $S^1 \times D^2$

§1.2 Structure theorems for 3-manifolds

Thm I (Heegaard splittings). Every closed, connected, orientable 3-manifold M has a Heegaard splitting, i.e. $\exists$ handle bodies V, W of genus g w/ $\partial V = \partial W = \Sigma_g$, $\exists\, \varphi.\ \partial V \xrightarrow[\text{homeo (or.-reversing)}]{\cong_{C^0}} \partial W$

s.t. $M = V \cup_\varphi W = V \sqcup W / p \sim \varphi(p)$, $p \in \partial V$, $\varphi(p) \in \partial W$

started w/ PhD thesis of the Danish mathematician Poul Heegaard

- Pf by Moise (1952) via triangulations
- we'll do: pf using handle decompositions

Ex: (1) $S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\}$

$H := \{(x,y,z,w) \mid w = 0\}$ hyperplane

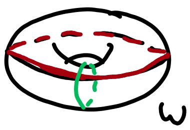
$H \cap S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid \underbrace{x^2+y^2+z^2 = 1, w=0}_{\|\cdot\| = 1}\} \cong S^2$

Exer: H splits S^3 into two balls $B_1 = \{(x,y,z,w) \mid \|\ \| = 1, w \geq 0\}$, $B_2 = \{ \quad \mid \quad w \leq 0\}$

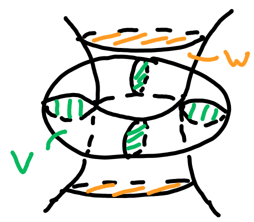
$\rightsquigarrow$ genus 0 Heegaard splitting of S^3

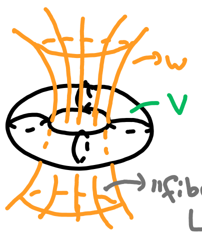
Note that

$D^2 \sqcup D^2/\sim\ = S^2$

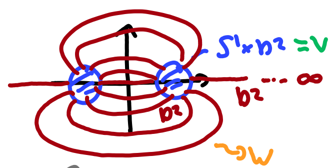
(2)  $\cup_\varphi$  $\cong S^3 \subseteq \mathbb{C}^2$

Picture in $\mathbb{R}^3$.





Side view:



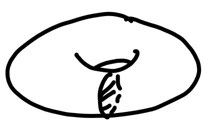
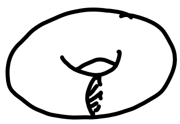
disks D^2 here correspond to orange disks there

Exer: $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 \leq \frac{1}{2},\ |z|^2 + |w|^2 = 1\} = V$

$\qquad |w|^2 \leq \frac{1}{2} \qquad = W$

$V \cap W = \{(z,w) \mid |z|^2 = |w|^2 = \frac{1}{2}\} \cong S^1 \times S^1$

(3) $S^1 \times S^2 = (S^1 \times D^2 \sqcup S^1 \times D^2) / (\theta_1, \theta_2) \sim (\theta_1, \theta_2)$

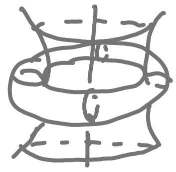
$D^2 \qquad D^2 \quad \rightsquigarrow S^2$

Thm II (Dehn surgery): Every closed, connected, oriented 3-mfd M
(Lickorish-Wallace)
can be obtained by Dehn surgery on S^3:

- Remove finitely many $S^1 \times D^2$ from S^3
- Reglue $S^1 \times D^2$'s via homeomorphisms $\partial(S^1 \times D^2) \to \partial(\overline{S^3 \setminus S^1 \times D^2})$

Ex:

- $(\underset{\mathbb{C}^2}{\overset{\wedge|}{S^3}} \setminus \underbrace{(D^2 \times S^1)}_{\subseteq \mathbb{C}^2}) \cup_\varphi (S^1 \times D^2) = (S^1 \times D^2 \sqcup S^1 \times D^2)/_\sim = S^1 \times S^2$



- $S^3 \setminus (S^1 \times D^2) \cup_{\varphi'} (S^1 \times D^2) = S^3$

Note that removing a solid torus from S^3 is the same as removing a tubular nbhd of a knot in S^3.

Rem: There are strong ties between understanding 3-mfds & understanding the embedded 1- & 2-submfds therein.
(knots & surfaces)

Thm III (Prime decomposition theorem)

We didn't have enough time to give the precise statement in class, but it is like the prime decomp for natural numbers a decomposition of 3-mfds into simpler, "prime" pieces using the connected sum operation, an analogue of conn. sum of surfaces (see above)

Outlook: For prime decomp, we split 3-mfds along spheres, can also split along tori. $\to$ JSJ decomp.,
Geometrization Conj.
This led to the pf of the famous Poincaré conjecture.
[Perelman; building on work on Ricci flow by Hamilton, ...]