Lecture III

Last week:

n-dim <u>k-handle</u> $D^k \times D^{n-k}$ attached to M^n along $\varphi: \partial D^k \times D^{n-k} \hookrightarrow \partial M$

Ex: 3-dim. 1-handle

Today:

- repeat pf of lemma w/ more details
- existence of handle decomp. & some facts about them from Morse theory

Def: $\varphi_0, \varphi_1 : Y \hookrightarrow X$ embeddings are isotopic if $\exists$ smooth homotopy $h: Y \times I \to X$ through embeddings, i.e. $h_t: Y \to X$ is an emb. $\forall\ 0 \le t \le 1$.

$h_t(\cdot) = h(\cdot, t)$, $h_0 = \varphi_0$, $h_1 = \varphi_1$

Thm (Isotopy Extension Theorem) (Thom, Cerf, Palais)

[Wall, Thm 2.4.2] [Kosinsky, Thm 5.2 in Chapter II]
[Hirsch, §8.1]

[there is a top. (locally flat) version due to Edwards–Kirby]

Y, X compact & $h: Y \times I \to X$ isotopy of embeddings $Y \hookrightarrow X$

$\Rightarrow \exists\ H: X \times I \to X$ smooth map s.t.

$$\left.\begin{array}{l} \bullet\ H_0 = id_X \\ \bullet\ H_t : X \to X \quad (H_t(\cdot) = H(\cdot, t)) \text{ diffeo } \forall 0 \le t \le 1 \\ \bullet\ h_t = H_t \circ h_0 \quad \forall t \end{array}\right\} \text{ambient isotopy}$$

[or diffeotopy]

[Pf idea: vector fields]

Lemma: $\varphi_i : \partial D^k \times D^{n-k} \hookrightarrow \partial M$, $i = 1,2$ isotopic embeddings

[Gompf–Stipsicz, p. 99f.]

$$\Rightarrow M \cup_{\varphi_1} h^k \cong_{C^\infty} M \cup_{\varphi_2} h^k.$$

Pf: Extend an isotopy h between φ_0, φ_1 to an ambient isotopy [$\rightsquigarrow h_0 = \varphi_0$, $h_1 = \varphi_1$]

$H: \partial M \times I \to \partial M$ using the Isotopy Extension Theorem.

Use this to find a diffeo $M_0 \dashrightarrow M_1$:

$$\begin{tikzcd} M_0 \arrow[d, dashed] & = & M \arrow[d, dashed, "H_1"] & \cup_{\varphi_0} & h^k \arrow[d, "id"] \\ M_1 & = & M & \cup_{\varphi_1} & h^k \end{tikzcd}$$

- $H_t : \partial M \to \partial M$ diffeo $\forall 0 \le t \le 1$
- $H_0 = Id_{\partial M}$
- $h_t = H_t \circ h_0\ \forall t$

In part., $H_1 \circ h_0 = H_1 \circ \varphi_0 = h_1 = \varphi_1$

To make this more precise, consider the diffeo $H \times Id_I : \partial M \times I \to \partial M \times I$

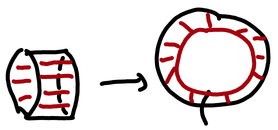
Now, identify $\partial M \times I$ w/ a collar neighborhood of ∂M in M.

Lemma (Collar nbhd theorem): For every mfd M w/ boundary ∂M, the bdry has a collar nbhd, i.e. $\exists$ an embedding $\chi : \partial M \times I \hookrightarrow \partial M$ which extends $\partial M \times \{0\} \to \partial M$, $(x, 0) \mapsto x$ $(\chi^{-1}(\partial M) = \partial M \times \{0\})$. ($\chi(\partial M \times I) \subset M$ closed.)

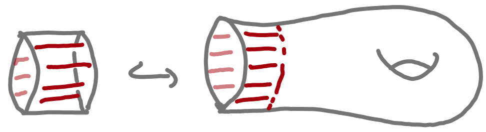
[Wall, Differential Topology, Thm 1.5.5] [Pf idea: vector fields]

[collar nbhds are unique up to diffeo [Wall 2.5.7], [Friedl, Alg. Top. Notes, 4.7.1]]

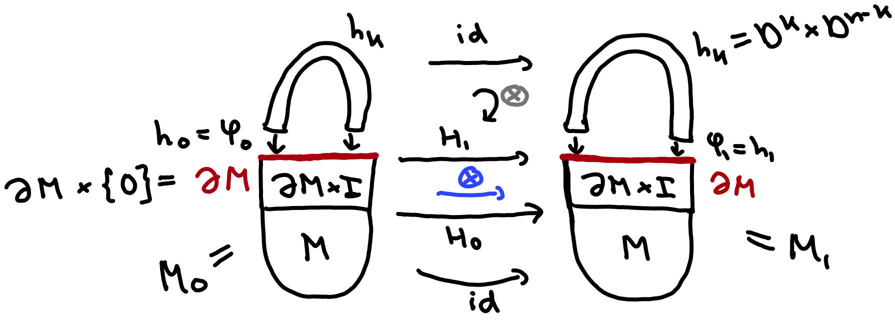
Ex: 1) M^n closed. $\begin{aligned} S^{k-1}\times M\times[0,1] &\to D^k\times M \\ (x,p,t) &\mapsto ((1-\tfrac{t}{2})x,p)\end{aligned}$ is a collar for $D^k\times M$.

2) 

$M=D^2$

3) 

Pf of above lemma continued: A diffeo $f: M_0\to M_1$ can be defined following the schematic below:



i.e.

$$f := \begin{cases} \text{id on } M\setminus\partial M\times I \\ \text{id on } D^k\times D^{n-k} \\ (x,t)\mapsto (H_{1-t}(x),t) \text{ on } \partial M\times I \; \otimes \end{cases}$$

$\otimes$ $H_1\circ \underbrace{h_0}_{=h_0} = \underbrace{h_1}_{=\varphi_1}\circ \mathrm{id}$

□

$\rightsquigarrow$ so need to specify the attaching map $\varphi: \partial D^k\times D^{n-k}\hookrightarrow \partial M$ only up to isotopy.

Rem: The isotopy class of $\varphi: \partial D^k\times D^{n-k}\hookrightarrow\partial M$ is determined by an embedding $\varphi_0: \underbrace{\partial D^k}_{=S^{k-1}}\times\{0\}\hookrightarrow\partial M$ $(=\varphi|_{\partial D^k\times\{0\}})$

together w/ a framing of $\varphi_0(S^{k-1})\subset\partial M$,

identification of the normal bundle of $\operatorname{Im}\varphi_0$ w/ $\partial D^k\times\mathbb{R}^{n-k}$, i.e. of $\nu(\varphi_0(S^{k-1}))$ w/ $S^{k-1}\times\mathbb{R}^{n-k}$

which boils down to a map $\varphi_0(S^{k-1})\to GL_{n-k}(\mathbb{R})$.

[comparing f w/ framing f_0 from trivial normal bundle, obtain element of GL_{n-k} [$\Leftrightarrow$ choice of basis of $T_p\mathbb{R}^{n-k}$]

$\rightsquigarrow$ {iso classes of framings of S^{k-1}} $\cong$ $\pi_{k-1}(GL(n-k)) \cong \pi_{k-1}(O(n-k))$]

Gram-Schmidt

Thm (Existence of handle decomp.):

Every closed (smooth) mfd M admits a handle decomposition, i.e. can be obtained from $\emptyset$ by finitely many handle attachments.

Rem: This also works for mfds w/ nonempty bdry.

Pf idea: M^n closed (smooth) mfd $\Longrightarrow$ (Sard's Thm, … [see e.g. Thm 2.20 in Matsumoto's book]) $\exists$ Morse function on M which is a smooth map $h: M \to \mathbb{R}$ w/ only non-degenerate critical points which are $(x_1, \dots, x_n) \mapsto -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$ in local coordinates.
(appropriate charts)
[Morse lemma, e.g. Thm 2.16, Matsumoto]

$\to$ critical pt of index k
[= nr. of neg. eigenvalues of $\mathrm{Hess}(f \circ \varphi^{-1})_{\varphi(p)}$]

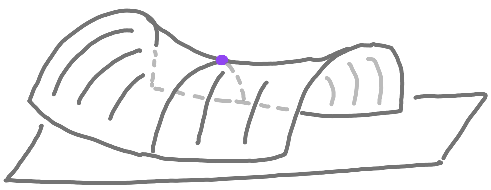
[Critical pt: $p \in M$ s.t. $\exists$ chart φ at p s.t. $\varphi(p) = 0$ & 0 is a critical pt of $h \circ \varphi^{-1}$ $[d(h \circ \varphi^{-1})_{\varphi(p)} = 0]$

$\hookrightarrow$ nondegenerate if $\det(\underbrace{\mathrm{Hess}((h \circ \varphi^{-1})_{\varphi(p)})}_{= \frac{\partial^2 h}{\partial x_i \partial x_j}}) \neq 0$]

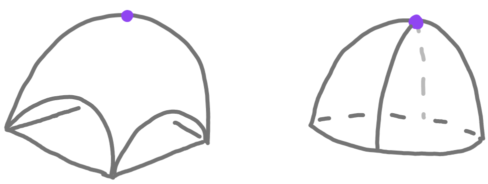
$$\boxed{\begin{array}{l} \text{passing through critical pts of } h \text{ of index } k \\ \qquad \longleftrightarrow \text{ attachment of } k\text{-handles} \end{array}}$$

Ex: graph of $x_1^2 + x_2^2$
(index 0, $n=2$)

graph of $x_1^2 - x_2^2$
(index 1, $n=2$)



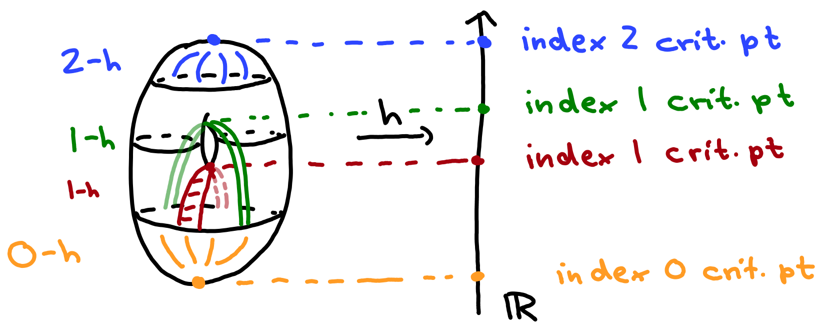
graph of $-x_1^2 + x_2^2$
(index 2, $n=2$)

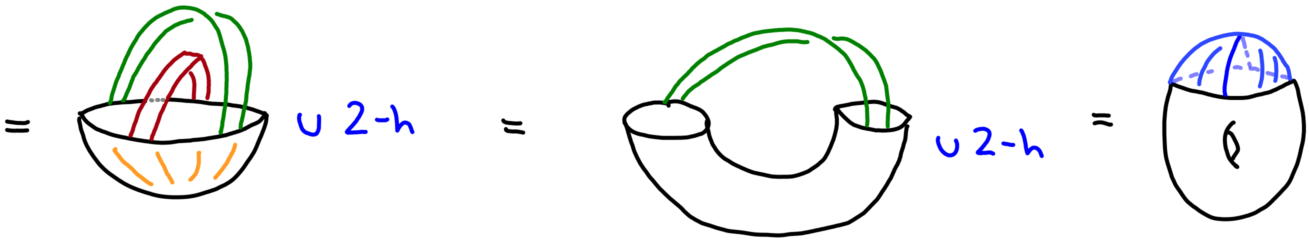


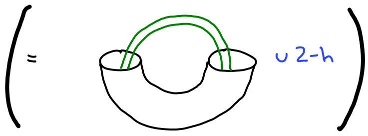
Some references:
- Matsumoto, An introduction to Morse theory
- Milnor, Lectures on the h-cobordism theorem
  ($\uparrow$ pf due to Smale)
- Wall, Differential topology

$\square$

Ex: $T^2 = S^1 \times S^1$, height fct $h: T^2 \to \mathbb{R}$ (proj. onto z-coord.) is a Morse function







We won't go into the details of Morse theory, but it has several nice consequences for us. We will only give rough ideas & pf sketches for the next lemmas concerning handle decompositions. If you want to know more details, have a look at the above references

Matsumoto, Milnor, Wall.

Some facts about handle decompositions (from Morse theory)

Lemma 1 (Reordering lemma): [Wall, 5.2.1] [Gompf-Stipsicz 4.2.7]

For $\ell \leq k$, $(M \cup_{\varphi_1} h_k) \cup_{\varphi_2} h_\ell \cong_{C^\infty} (M \cup_{\varphi_2''} h_\ell) \cup_{\varphi_1'} h_k$.

(by isotoping att. maps; $\varphi_2'' \underset{iso}{\cong} \varphi_2$, φ_1' has same image as φ_1).

Consequence: Handles can be attached in increasing order (first 0-handles, then 1-h, 2-h, ...).

We can also think of handles of the same index being attached simultaneously.

Sketch of pf of Lemma 1: $\quad A_e :=$ att. sphere of h_e
$$\overset{\cong}{S^{e-1}} \subset \partial(M \cup h_k)$$
$$B_k := \text{belt sphere of } h_k$$
$$\overset{\cong}{S^{n-k-1}} \subset \partial(M \cup h_k)$$

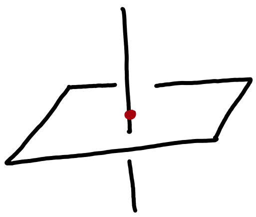
Note that $\dim(A_e) + \dim(B_k) = e-1+n-k-1 = n + \underbrace{e-k}_{\leq 0} - 2 < n-1 = \dim(\partial(M\cup h_k))$.

Interlude on transversality:

Def: Two (embedded) submfds Y, Z of X intersect transversely if $\forall p \in Y \cap Z$ the tangent spaces T_pY & T_pZ together span T_pX. Write $Y \pitchfork Z$.

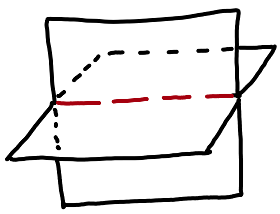
For $Y^m, Z^n \subset X^k$ w/ $m+n < k$, transv. intersection means $Y \cap Z = \emptyset$.

Ex: 1)



local picture of a transverse intersection of a 1- & a 2-submfd of a 3-mfd.

2)



local picture of a transverse intersection of two 2-submfds in a 3-mfd.

Sketch of pf continued:

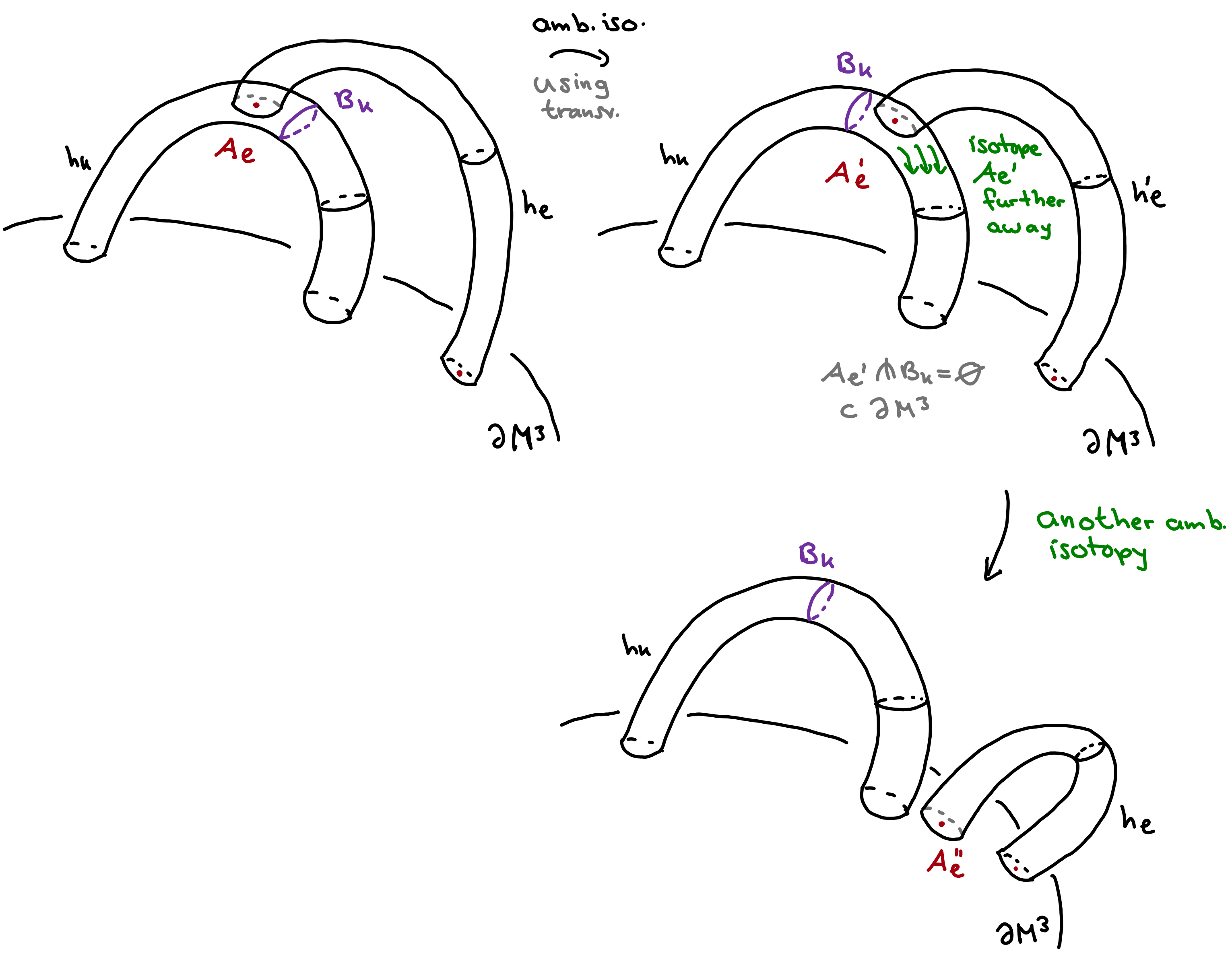
Transversality thm (Thom) $\Rightarrow$ There is an ambient isotopy of $\partial(M\cup h_k)$ taking A_e to A_e' s.t. $A_e' \pitchfork B_k$.

By the calculation of dimensions above, this implies $A_e' \cap B_{k-1} = \emptyset$ $\subset \partial(M \cup h_k)$.

[The transversality thm implies that given two embeddings $f, g : Y, Z \hookrightarrow X$, we can perturb f by an ("arb. small") isotopy to $f' : Y \hookrightarrow X$ s.t. $f'(Y) \pitchfork g(Z) \subset X$. [see e.g. Chapter 4.5 in [Wall], Chapter 6 in [Lee].]]

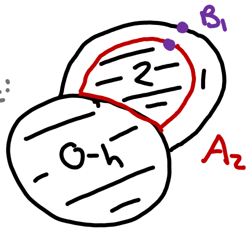
We can now isotope further ($\exists$ amb. isotopy s.t. ...) so that $A_e'' \subset \partial M$ (away from h_k) & even $\varphi''(S^{e-1} \times D^{n-e}) \subset \partial M$.
[use e.g. vector field directed radially away from cocore of h_k or see Wall, 5.2.1 for details] $\square$

Ex illustrating above pf strategy of Lemma 1: $n=3,\ k=\ell=1$

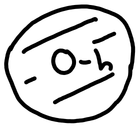


Handle cancellation

Ex: 1) $n=2$:



$\cong_{C^\infty}$



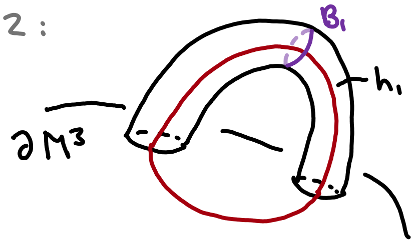
1-h is $D^1 \times D^1$ att. along $\partial D^1 \times D^1$

belt sphere $B_1 = \{0\} \times \partial D^1$

2-h is $D^2 \times \{0\}$

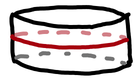
att. sphere $A_2 = \partial D^2 \times \{0\}$

2) $n=3, k=2$:



attach $h_2 = D^2 \times D^1$

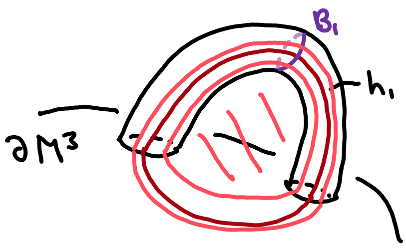
along $S^1 \times D^1$



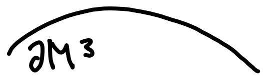
att. sph. $A_2 = \partial D^2 \times \{0\}$

$B_1 = \{0\} \times \partial D^2 = \{0\} \times S^1$

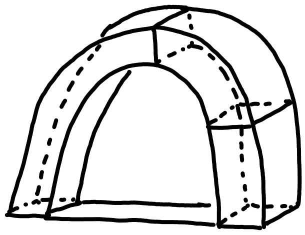
$\rightsquigarrow$



$\cong_{C^\infty}$



Better picture:



Lemma 2 (Handle cancellation lemma):

[Wall, 5.4.3]
[Milnor, 5.4]
[Gompf-Stipsicz, 4.2.9]
[Matsumoto, 3.34]

If $A_k \pitchfork B_{k-1} = \{pt\}$,

then $(M \cup h_{k-1}) \cup h_k \cong_{C^\infty} M$,

where $A_k =$ att. sphere of h_k

$B_{k-1} =$ belt sphere of h_{k-1}.

Observation (**Dual handle decomposition**):

Every handle decomposition can be turned "upside down", i.e.

$$M^n = \emptyset \cup_{\varphi_1} h_1^{k_1} \cup_{\varphi_2} h_2^{k_2} \cup \dots \cup_{\varphi_m} h^{k_m}$$

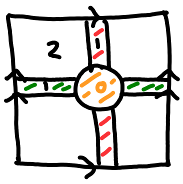
$$\cong_{C^\infty} \emptyset \cup_{\varphi_m} \tilde{h}^{n-k_m} \cup \dots \cup_{\varphi_1} \tilde{h}^{n-k_1}$$

where $h^{k_e} = D^{k_e} \times D^{n-k_e} \cong D^{n-k_e} \times D^{k_e} \cong \tilde{h}^{n-k_e}$.

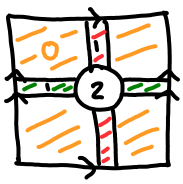
The roles of core and cocore, belt & att. region are reversed.

In Morse theory, this corresponds to $h \rightsquigarrow -h$.

Ex:



$\cong_{C^\infty}$



Prop: M^n connected, closed $\Rightarrow$ $\exists$ handle decomposition

[Gompf-Stipsicz, 4.2.13] w/ • exactly one 0-handle

[Matsumoto, 3.35] • –"– –"– n-handle.

Pf: Take any handle decomp. of M.

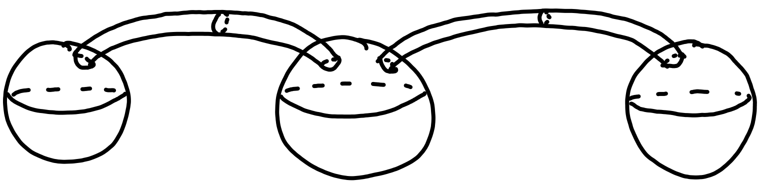
M closed $\Rightarrow$ $\exists$ at least one 0-handle.

[only type of handle we can attach to $\emptyset$]

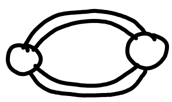
Let $h_0^1, h_0^2, \dots, h_0^n$ be the 0-handles.

M connected $\Rightarrow$ h_0^i are connected by 1-handles.

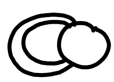
(att. spheres of h_k for $k \geq 2$ are connected and thus must be attached to a single conn. comp.)



We can cancel pairs of 0-h & 1-h. e.g.



$\cong$



$\Rightarrow \exists!$ 0-h.

(belt sphere of 0-h $B_0 \cong S^{n-1}$

att. sph. of 1-h $A_1 \cong S^0$)

Dual handle decomp. $\Rightarrow \exists!$ n-handle. $\square$