Lecture V

Last week: Every closed, connected, orientable 3-mfd M^3 has a Heegaard splitting $H_1 \cup_\varphi H_2$.
$\uparrow$ $\uparrow$ handle bodies of genus $g = g(\partial H_1) = g(\partial H_2)$.

Prop 1: $H_i \cong_{C^\infty} \natural_g (S^1 \times D^2)$, $i = 1, 2$.

Pf: In notes for Lecture 4. [Orientability + framing]

A Heegaard splitting of M^3 of genus g determines a **planar Heegaard diagram** for M, which is

$\mathbb{R}^2$ together w/
- g pairs of disks $D^2_{i,1} \sqcup D^2_{i,2}$, $i = 1, \dots, g$,
- curves / arcs $\beta_j \subset \mathbb{R}^2 \setminus \operatorname{Int}(D^2_{i,1} \sqcup D^2_{i,2})$, $i = 1, \dots, g$,
  [arcs have endpts on the disks]

not necessarily g arcs

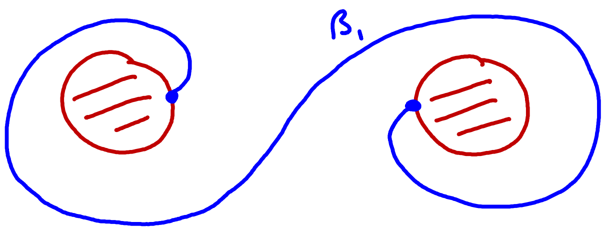
which become g att. circles $\mu_1, \dots, \mu_g$ in ∂H_1
(after identifying $D^2_{i,1}$ w/ $D^2_{i,2}$ $\forall i$)

Equivalently, $(\underset{\substack{\| \\ \partial H_1}}{\Sigma_g}, \underbrace{\mu_1, \dots, \mu_g})$ is a **Heeg. diagram.**
(or.) simple closed curves on Σ_g

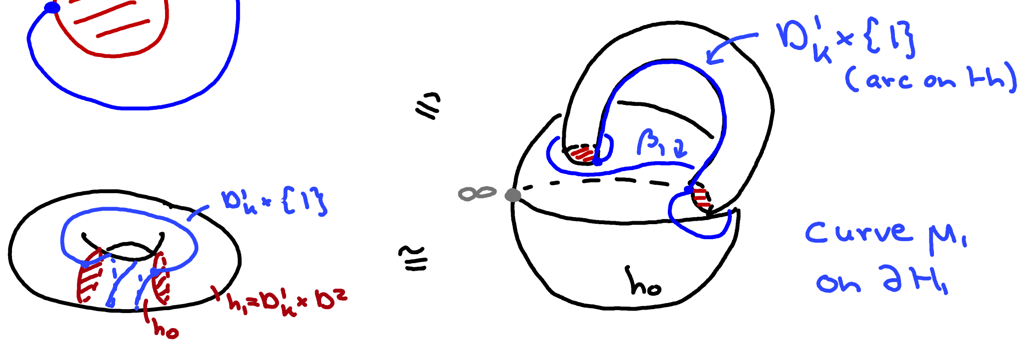
$\mu_i = \beta_i \cup_k \overbrace{(D^1_k \times \{1\})}^{\text{arc on } k\text{-th 1-handle } D^1_k \times D^2 \text{ corresponding to identification of the disks } D^2_{k,1} \text{ \& } D^2_{k,2}}$

$\uparrow$
$\forall k$ s.t. β_i meets $D^2_{k,1}$ & $D^2_{k,2}$

Ex:



$\subset \mathbb{R}^2 \subset \partial h_0 = \partial D^3 = S^2 \cong \mathbb{R}^2 \cup \{\infty\}$



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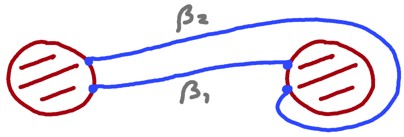
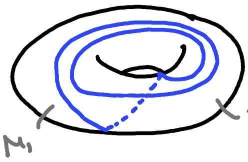


$\hat{=}$



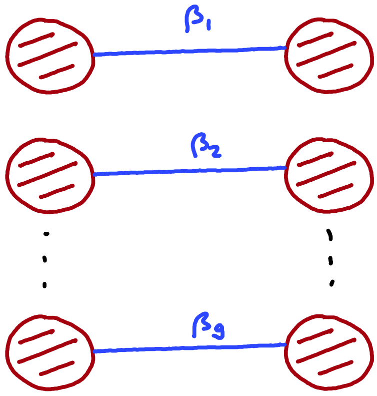
$\partial H_1 \cong \Sigma_1 = S^1 \times S^1$

[later: $\rightsquigarrow L_{2,1} \cong \mathbb{R}P^3$]

this shows that there are not necessarily g arcs β_j

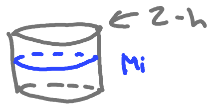
(but there should always be g curves on $\underset{\Sigma_g}{\overset{\partial H_1}{\wr\wr}}$; here $g=1$)

Ex:



genus g Heegaard diagram for S^3

$\forall \mu_i$ we attach a 2-handle along attaching sphere μ_i

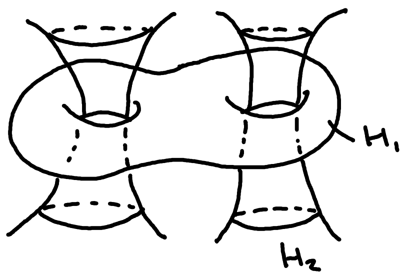


$(\mu_i \cong \partial D^2 \times \{0\} \subset \underbrace{D^2 \times D^1}_{\text{2-h.}})$

this "fills the holes" of H_1 $\rightsquigarrow$ get D^3;

glue $h_3 \cong D^3 \rightsquigarrow S^3$

$g=2$:



$S^3 \setminus (\natural_g (S^1 \times D^2)) = H_2$

$\cong \natural_g (S^1 \times D^2)$

Observations about Heegaard diagrams:

Let $M = H_1 \cup H_2$ be a Heegaard splitting & let $(\partial H_1, \mu_1, \dots, \mu_g)$ be the corresponding Heegaard diagram. Then

(1) • $\mu_1, \dots, \mu_g \subseteq \partial H_1$ are pairwise disjoint (=nonintersecting)

(at least we can assume them to be disjoint by reordering lemma $\rightsquigarrow$ 2-h. attached simultaneously)

(3) • $\mu_1, \dots, \mu_g$ bound disks in H_2

(the cores $D^2 \times \{0\}$ of the 2-handles $D^2 \times D^1$ att. along μ_i)

(2) • $\partial H_1 \setminus (\mu_1 \cup \dots \cup \mu_g)$ is connected

$\left(\text{use Euler} \overset{}{\Longleftrightarrow} \text{characteristic } \chi \quad \partial(H_1 \cup \{h_2^i \text{ att. along } \mu_i\}_{i=1}^{g} \cong S^2 \right)$

["$\Rightarrow$": $S := \partial(H_1 \cup \{\text{2-h. att. along } \mu_i\})$, $\partial H_1 \cong \Sigma_g$

$\chi(S) = \chi(\partial H_1) + 2g = 2 - 2g + 2g = 2 \xRightarrow{S \text{ conn.}} S \cong S^2$

b/c S is obtained from ∂H_1 by adding g pairs of disks and removing g annuli $\mu_i \times I$; $\chi(D^2) = 1$, $\chi(S^1 \times I) = \chi(S^1) = 0$

See also V. Prasolov and A. Sossinsky, Knots, Links, Braids and 3-Manifolds: An Introduction to the New Invariants in Low-Dimensional Topology, p. 77]

[(2) $\overset{\text{harder}}{\Longleftrightarrow}$ $[\mu_1], \dots, [\mu_g] \in H_1(\partial H_1; \mathbb{Z})$ are linearly independent.

assuming (1). first homology]

Conversely, we have:

to match w/ numbering of Lecture 4 [Thm 2 in §3]

Thm 2 (From Heegaard diagrams to Heegaard splittings):

Let $\mathcal{H} = (\partial H_1, \mu_1, \dots, \mu_g)$ be a Heegaard diagram s.t. (1) & (2) are satisfied. Then $\mathcal{H}$ determines a (Heegaard splitting of a) unique closed, or. 3-mfd M^3 up to diffeomorphism.

Sketch of proof: (More details e.g. in the books by Matsumoto, (Thm 5.14) Prasolov – Sossinsky, (§8 & §10) Gompf – Stipsicz [p. 99–102, p. 112f.])

Step 1: ∂H_1 determines H_1 (by Prop. 1).

Step 2: Consider the attaching map of a 2-handle, i.e.

$\varphi_i : \partial D^2 \times D^1 \hookrightarrow \partial H_1$.

$\mathcal{H}$ determines $\varphi_{i,0}(\overbrace{\partial D^2 \times \{0\}}^{\text{attaching sphere of the 2-handle}}) \overset{!}{=} \mu_i \subset \partial H_1$,

where $\varphi_{i,0} := \varphi_i|_{\partial D^2 \times \{0\}}$.

Fact: Up to isotopy, $\exists!$ embedding φ_i extending $\varphi_{i,0}$ which makes $H_1 \cup h_2^i$ orientable.

[Similarly to the pf of Prop. 1 (see Lect. 4) this follows from considering framings]

⌈ A priori, we have two choices, which come from framings of $\varphi_{i,0}(\partial D^2 \times \{0\}) \cong S^1$,

$$\{\text{framings of } S^1\}/_{\text{isotopy}} = \{S^1 \mapsto GL_1(\mathbb{R}) = \mathbb{R} \setminus \{0\}\}/_{\text{iso}}$$
$$= \pi_1(GL_1(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$$ ⌋

We saw: Isotopic att. maps $\overset{\text{Lemma Lect. 2/3}}{\Longrightarrow}$ diffeomorphic mfds.

$\Rightarrow$ $\mathcal{H}$ determines $H_1 \cup \{2\text{-handles } h_2^i\}_{i=1}^{g} =: M_2$ (up to diffeo).

Step 3: To obtain M^3, we need to attach a 3-handle h^3 along $\varphi : \partial D^3 \times \{0\} \hookrightarrow \partial M_2 \underset{\substack{\uparrow \\ \text{b/c of (2)}}}{\cong} S^2$.

But for a different choice of $\varphi' : \partial D^3 \times \{0\} \hookrightarrow \partial M_2$,

$\varphi'^{-1} \circ \varphi : \partial D^3 \to \partial D^3$ <u>extends</u> to a diffeo $D^3 \to D^3$.

[Smale '59, "Diffeomorphisms of the 2-sphere"] $\square$

<u>Exer</u> : (Weaker version of) "<u>Alexander trick</u>" [Exercise Sheet 2, Exer 4]

Every <u>homeomorphism</u> $\overset{f:\ \partial D^n \to \partial D^n}{f : \partial D^3 \to \partial D^3}$ extends to a <u>homeo</u> $\overset{D^n \to D^n}{D^3 \to D^3}$.

In fact, this is true $\forall n$, here $n = 3$.

Rem: Smoothly (C^∞/ diffeos) true for $n=1,2,3$, but not e.g. $n=7$ (Milnor). [see also Gompf - Stipsicz, p.102]

Rem. (*): A variation of the above proof [see Matsumoto] shows that also $(\partial H_1, h_1(\mu_1), \ldots, h_1(\mu_g))$ for h_t an isotopy of ∂H_1 w/ $h_0 = id$ also gives M up to diffeo.
($\rightsquigarrow$ $h_1 : \partial H_1 \xrightarrow[C^\infty]{\cong} \partial H_1$)

We have seen : $\{$Heegaard diagrams$\}$ $\overset{\text{Thm 2}}{\underset{\text{today}}{\twoheadrightarrow}}$ $\{$Heegaard splitting$\}$
(see above)

"$\twoheadrightarrow$" = surjective map

$\downarrow$ Thm 1 (Lecture 4)

$\{$or., cld, conn. 3-mfds$\}$

Q: What about the "kernels" of these maps?
(not very precise question / maps!)

Thm 3 (Reidemeister - Singer 1933):

Two Heegaard diagrams give rise to the same or., cld, conn. M^3 (up to diffeo) if they are related by

- isotopies of the attaching spheres $\mu_1, \ldots, \mu_g$ (as in Rem. (*))
- handle slides
- finitely many (de-) stabilizations. $\rightsquigarrow$ will explain below what these are

The original pf of this thm was in the PL category.
The pf in the smooth category (i.e. for smooth 3-mfds as in our setting) uses Cerf theory and was written up by Laudenbach in 2014
↳ Cerf 1970's ↳ "A proof of Reidemeister-Singer's Theorem by Cerf's methods"
(built on lost notes by Bonahon from the 1970's).

We won't prove Thm 3, but explain handle slides & stabilizations a bit more below.

Using Thm 3, a more accurate diagram (still not very precise) of the above maps would be:

$$\left.\{\text{Heegaard diagrams}\}\middle/\begin{matrix}\text{1- \& 2- handle slides}\\ \text{\& isotopies}\end{matrix}\right. \xrightarrow{1:1} \{\text{Heegaard splittings (up to } \underline{\text{equivalence}})\}$$

$\exists$ diffeo $h: M \to M$
s.t. $h: H_i \mapsto H_i'$
$\Sigma \mapsto \Sigma'$, $h|_\Sigma : \Sigma \overset{\cong_{C^\infty}}{\to} \Sigma'$ (or. pres.)

$$\left.\{\text{Heegaard splittings (up to equivalence)}\}\middle/\text{(de-) stabilizations}\right. \xrightarrow{1:1} \{\text{or., closed, conn. 3-mfds}\}$$

Stabilization

Motivation: $M^3 = \underbrace{(h_0 \cup h_1^1 \cup h_1^2 \cup \dots \cup h_1^g)}_{H_1} \cup_\Sigma \underbrace{(h_2^1 \cup h_2^2 \cup \dots \cup h_2^g \cup h_3)}_{H_2}$

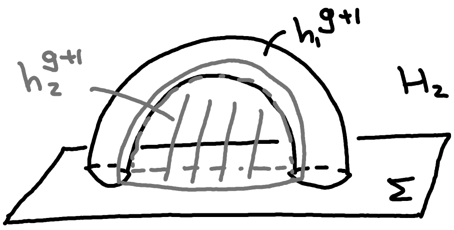
handle cancellation

$\cong_{C^\infty} \underbrace{(h_0 \cup h_1^1 \cup h_1^2 \cup \dots \cup h_1^g \cup h_1^{g+1})}_{H_1'} \cup_{\Sigma'} \underbrace{(h_2^{g+1} \cup h_2^1 \cup h_2^2 \cup \dots \cup h_2^g \cup h_3)}_{H_2'}$

if h_1^{g+1} & h_2^{g+1} cancel each other.

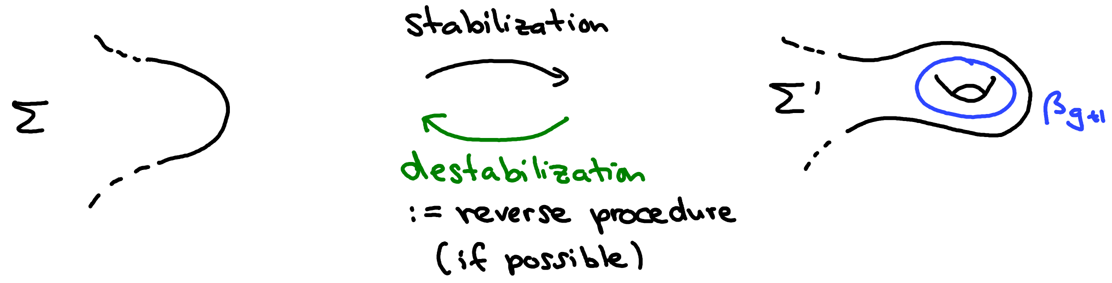
["introduce a cancelling pair of 1- & 2-handles"]

$\rightsquigarrow g(\Sigma') = g(\Sigma) + 1.$



For a Heegaard diagram $(\partial H_1, \mu_1, \dots, \mu_g)$, <u>stabilization</u> means taking connected sum w/ $(S^1 \times S^1, S^1 \times \{pt\})$, i.e.

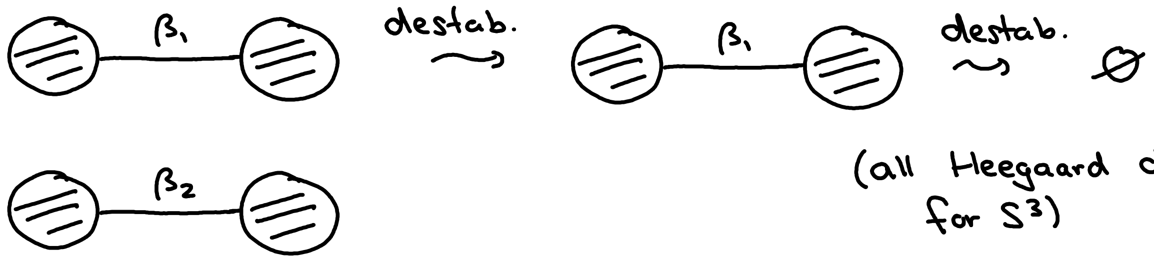
say $1 \in S^1$



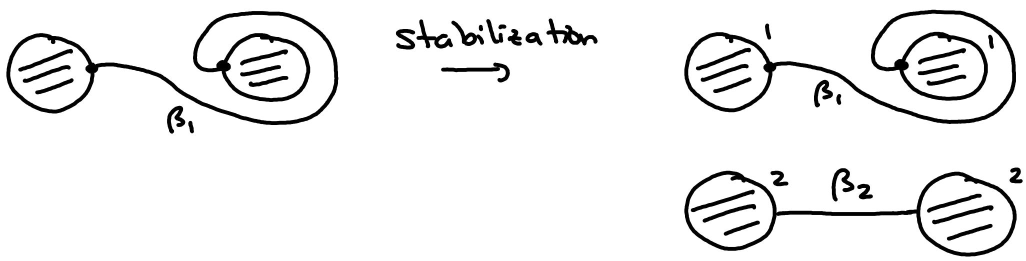
[Note that 

is a Heegaard diagram for S^3.

On the level of 3-mfds you could also view the result of a stabilization be given by $(M, \partial H_1) \# (S^3, S^1 \times S^1) = (\underbrace{M \# S^3}_{\cong M}, \partial H_1 \# S^1 \times S^1)$ where $\#$ denotes connected sum of mfds [which we haven't def'd yet.]]

Ex: 

(all Heegaard diagrams for S^3)

Ex: 

Handle slides

[See e.g. Thm 3.2.1 in Matsumoto or Def. 4.2.10 in Gompf-Stipsicz.]

Rem: We've encountered the idea of handle sliding already when thinking about handle cancellations in handle decompositions. (in sketch of pf why $M \cup h_k \cup h_{k+1} \cong_{C^\infty} M$ for cancelling pair h_k, h_{k+1})

Idea: Given handles h_k^1, h_k^2 of the same index k, both attached to $\partial(M^n)$, i.e. $h_k^1 \cong h_k^2 \cong_{C^\infty} D^k \times D^{n-k}$.

Move A_k^1, the attaching sphere of h_k^1 across B_k^2, the belt sphere of h_k^2 through an isotopy in $\partial(M \cup h_k^2)$.

Because $\dim A_k^1 = k-1$, $\dim B_k^2 = n-k-1$ and $k-1+n-k-1 = n-2$, $n-2 < n-1 = \dim(\partial(M \cup h_k^2))$, at an intermediate step of this isotopy $(A_k^1)'$ & $(B_k^2)'$ will intersect in one point

where $T_p(A_k^1)' \oplus T_p(B_k^2)'$ has codimension 1 in $T_p(\partial(M \cup h_k^2))$.

So we have 2 choices of directions for pushing (A_k^1) off $(B_k^2)'$:

go back to A_k^1 & B_k^2 or obtain $(A_k^1)''$ & $(B_k^2)''$.

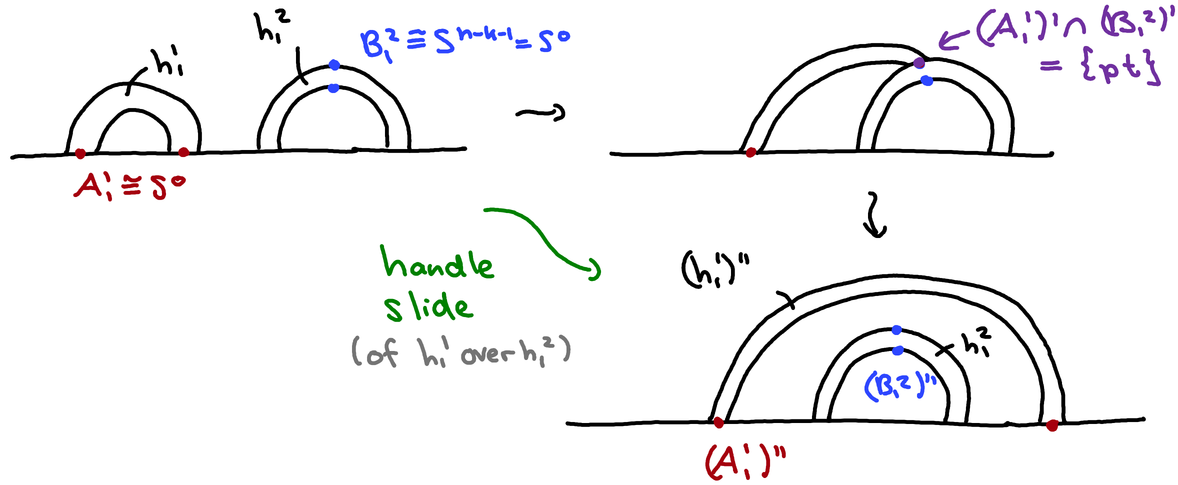
The result of the handle slide applied to $M \cup h_k^2 \cup h_k^1$

now is $M \cup \underbrace{h_k^2}_{=(h_k^2)''} \cup (h_k^1)''$ where $(h_k^1)''$ is attached to $M(\cup h_k^2)$

along the attaching sphere $(A_k^1)''$. $\rightsquigarrow$ "Handle slide of h_k^1 over h_k^2"

[More precisely, let $\varphi_i : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ be the attaching maps of $h_k^{1,2}$, $i=1,2$. Take an isotopy h_t of ∂M describing the above process (moving A_k^1 "through" B_k^2) s.t. $h_0 = \mathrm{id}_{\partial M}$ and use $h_1 \circ \varphi_i$ as attaching maps for $(h_k^1)''$ & $(h_k^2)''$.]

Ex: $\underline{n=2}$: $\underline{k=1}$

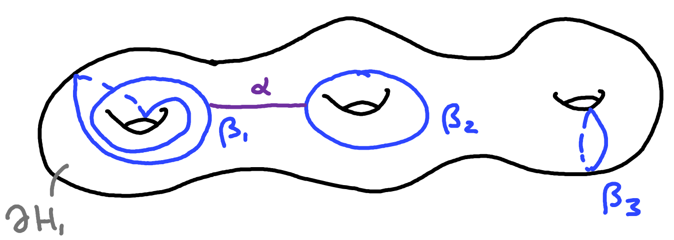


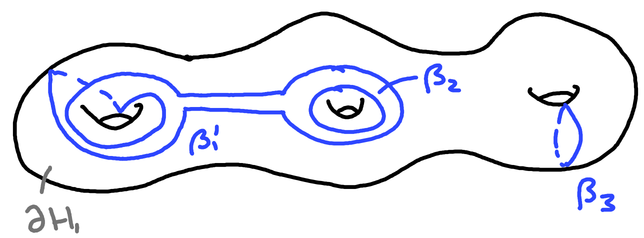
handle slide (of h_1^1 over h_1^2)

What is a handle slide in a Heegaard diagram?

2-handle slide in a Heegaard diagram:

Ex:

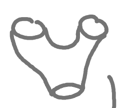




Choose arc α disjoint from β_1 & β_2 except at its endpoints, one endpt on β_1, one on β_2, α disjoint from $\beta_3, \dots, \beta_g$.

β_1' = "connected sum of β_1 & β_2 along α"

($\Leftrightarrow$ $\beta_1, \beta_2, \beta_1'$ bound embedded pair of pants in $\partial H_1 \setminus (\beta_3 \cup \dots \cup \beta_g)$)



Ex: